

Nonequilibrium Renormalization Theory I

D.V. Prokhorenko

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Abstract

In the present article we consider some general class of divergent diagrams in Keldysh diagram technique. These divergences arise for non-equilibrium matter and closely related to the divergences in the kinetic theory of gases. We suggest a scheme of renormalization of such divergences and illustrate it on some examples. In the other papers of these series we develop the general theory of renormalization of non-equilibrium diagram technique. The fact that thermal divergences in non-equilibrium diagram technique can be renormalized leads to the following consequence: to prove that the system tends to the thermal equilibrium one should to take into account the behavior of the system on its boundary. In this paper we illustrate this fact on Bogoliubov derivation of kinetic equations.

1 Introduction

There are exist divergences in virial decomposition of kinetic equations. This fact was observed by Cohen and Dorfman [1]. It is possible to summize some set of diagrams to obtain a finite result. It was done by Kawasaki and Oppenheim [2].

Our main goal in this series of papers is to analyze such divergences. In the present paper we study divergences in Keldysh diagram technique which arise if the state of the matter is non-equilibrium. It is more or less obvious that these divergences are the same as the divergences in the kinetic equations.

In the present series of papers we develop the general theory of such divergences. As result for a wide class of Bose systems in the sense of formal power series on coupling constant we find non-Gibbs state $\langle \cdot \rangle$ such that the correlators

$$\langle \Psi^\pm(t, x_1) \dots \Psi^\pm(t, x_n) \rangle \quad (1)$$

are translation invariant, do not depend on t and satisfy to the weak cluster property. Here Ψ^\pm are the fields operators and the weak cluster property means the following

$$\begin{aligned} \lim_{|a| \rightarrow \infty} \int \langle \Psi^\pm(t, x_1 + \delta_1 e_1 a) \dots \Psi^\pm(t, x_n + \delta_n e_1 a) \rangle f(x_1, \dots, x_n) d^3 x_1 \dots d^3 x_n \\ = \int \langle \Psi^\pm(t, x_{i_1}) \dots \Psi^\pm(t, x_{i_k}) \rangle \langle \Psi^\pm(t, x_{i_k}) \dots \Psi^\pm(t, x_{i_n}) \rangle \\ \times f(x_1, \dots, x_n) d^3 x_1 \dots d^3 x_n, \end{aligned} \quad (2)$$

there $\delta_i \in \{1, 0\}$, $i = 1, 2 \dots n$ and

$$\begin{aligned} i_1 < i_2 < \dots < i_k, \\ i_{k+1} < i_{k+2} < \dots < i_n, \\ \{i_1, i_2, \dots, i_k\} &= \{i = 1, 2 \dots n | \delta_i = 0\} \neq \emptyset, \\ \{i_{k+1}, i_{k+2}, \dots, i_n\} &= \{i = 1, 2 \dots n | \delta_i = 1\} \neq \emptyset. \end{aligned} \quad (3)$$

$f(x_1, \dots, x_n)$ is a test function, e_1 is a unite vector parallel to the x -axis. In the present paper we consider only a some wide class of divergent diagrams.

Let us prove that the existence of such states implies non-ergodic property of the system. We consider the problem only on classical level. Suppose that

our system is ergodic, i.e. there no first integrals for the system except energy. Then, the distribution function depends only on energy. We can represent the distribution function $f(E)$ as follows:

$$f(E) = \sum c_\alpha \delta(E - E_\alpha), \quad (4)$$

where the sum can be continuous (integral). Let 1 be some finite subsystem of our system. Let 2 be a subsystem obtained from 1 by translation on the vector L parallel to the x -axis of enough large length. Let 12 be a union of the subsystems 1 and 2. Let ρ_1 , ρ_2 and ρ_{12} be distribution functions for the subsystems 1, 2 and 12 respectively. Let Γ_1 , Γ_2 and Γ_{12} be points of the phase spaces for the subsystem 1, 2 and 12 respectively. By the same method as the method used for the derivation of the Gibbs distribution we find:

$$\rho_{12} = \sum c_\alpha \frac{e^{-\frac{E_{\Gamma_1}}{T}}}{Z_\alpha} \frac{e^{-\frac{E_{\Gamma_2}}{T}}}{Z_\alpha} \quad (5)$$

in the obvious notation. But the weak cluster property implies that

$$\rho_{12} = \rho_1 \rho_2. \quad (6)$$

Therefore all the coefficients c_α are equal to zero except one. We find that

$$f(E) = c \delta(E - E_0) \quad (7)$$

for some constants c and E_0 . So each finite subsystem of our system can be described by Gibbs formula and we obtain a contradiction.

Non-ergodic property means that there no thermalization in infinite Bose-gas system.

This fact implies that to prove that the system tends to thermal equilibrium we should to take into account the behavior of the system on its boundary. Indeed if the system has no boundary the system is infinite.

To illustrate this fact we will study Bogoliubov derivation of kinetic equations. When one derives BBGKI-chain one neglects by some boundary terms. If one take into account this boundary terms and use Bogoliubov method of derivation of kinetic equations one find that these boundary terms compensate the scattering integral.

The paper composed as follows. In section 2 we introduce the notions of the algebra of canonical commutative relations. In section 3 we describe our

model. In section 4 we describe non-equilibrium (Keldysh) diagram technique. In section 5 we discuss the divergences in our model and the method of its renormalizations. In section 6 we give a proof that there exists divergences in our theory. In section 7 we describe regularization which will be used. In section 8 we discuss some simple relation on Green functions. In section 9 we make renormalization procedure in one-chain approximation. In section 10 we begin to renormalize diagrams in two-chain approximation. We calculate divergent parts of all diagrams in this approximation. In section 11 we discuss subdivergences i.e. we calculate the contributions which comes from counterterms for the one-chain diagram. In section 12 we show that the divergent part of all diagrams which is proportional to $\frac{1}{\varepsilon^2}\delta(\omega - \omega(p))$ can be subtracted by counterterms. In section 13 we show that the divergent part of all diagram which is proportional to $\frac{1}{\varepsilon^2}\delta'(\omega - \omega(p))$ can be subtracted by counterterms. In section 14 we study Bogoliubov derivation of kinetic equations and show that the scattering integral is compensated by some boundary terms which are usually neglected. Section 15 is a conclusion.

2 The algebra of canonical commutative relations

The algebra of canonical commutative relations is a \star algebra with a unite generated by generators

$$a(f), a^+(f), \quad (8)$$

where f belongs to the Schwartz space of test functions $S(\mathbb{R}^3)$. The generators $a(f)$, $a^+(f)$ satisfies the following relations:

$a(f)$ is an antilinear functional on f ,
 $a^+(f)$ is a linear functional on f and

$$\begin{aligned} [a(f), a^+(g)] &= \langle f, g \rangle \\ [a(f), a(g)] &= [a^+(f), a^+(g)] = 0. \end{aligned} \quad (9)$$

Here $\langle f, g \rangle$ is a standard scalar product in $L^2(\mathbb{R}^3)$

$$\langle f, g \rangle = \int f^*(k)g(k)d^3k. \quad (10)$$

The involution \star is defined by the following rule

$$(a(f))^\star = a^+(f). \quad (11)$$

The "field operators" $\Psi(x)$ $\Psi^+(x)$ are defined by the following formulas

$$\begin{aligned} \Psi(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{ikx} a(k) d^3k, \\ \Psi^+(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{-ikx} a^+(k) d^3k. \end{aligned} \quad (12)$$

Here we have used the following formal notation

$$a^+(f) = \int a(k) f(k) d^3k. \quad (13)$$

Definition. Let us consider a Gauss state ρ_0 on the algebra of canonical commutative relations defined by its two-point correlator as follows

$$\begin{aligned} \rho_0(a(k)a(k')) &= \rho_0(a^+(k)a^+(k')) = 0, \\ \rho_0(a^+(k)a(k')) &= n(k)\delta(k - k'). \end{aligned} \quad (14)$$

If

$$n(k) = \frac{e^{-\beta\omega(k)}}{1 - e^{-\beta\omega(k)}}, \quad (15)$$

the state ρ_0 is called a Plank state. Here $\omega(k) = \frac{k^2}{2} - \mu$, $\mu < 0$.

3 The model

Our model is described by the following Hamiltonian

$$H = H_0 + \lambda V, \quad (16)$$

where

$$\begin{aligned} H_0 &= \int \omega(k) a^+(k) a(k) d^3k, \\ \omega(k) &= \frac{k^2}{2} - \mu, \quad \mu < 0, \end{aligned} \quad (17)$$

$$V = \frac{1}{2} \int \Psi^+(x) \Psi^+(x') V(x-x') \Psi(x') \Psi(x) d^3x d^3x',$$

$$\lambda \in \mathbb{R}, \quad (18)$$

V is an interaction, $V(x) \in S(\mathbb{R}^3)$. Let us rewrite the interaction in the Fourier representation

$$V = \frac{1}{2(2\pi)^3} \int d^3k_1 d^3k_2 d^3k'_1 d^3k'_2 \tilde{V}(k_1 + k_2) \times$$

$$\delta(k_1 + k_2 - k'_1 - k'_2) a^+(k'_1) a^+(k'_2) a(k_1) a(k_2). \quad (19)$$

Here by definition

$$\tilde{V}(k) = \int e^{ikx} V(x) d^3x. \quad (20)$$

4 Nonequilibrium diagram technique

Let us introduce the Green functions for the system

$$\rho(T(\Psi_H^\pm(t_1, x_1), \dots, \Psi_H^\pm(t_n, x_n))). \quad (21)$$

Here symbol H near Ψ^\pm means that Ψ_H^\pm are Heizenberg operators.

In nonequilibrium diagram technique we admit the following representation for the Green functions

$$\rho(T(\Psi_H^\pm(t_1, x_1), \dots, \Psi_H^\pm(t_n, x_n))) =$$

$$\rho(S^{-1} T(\Psi_0^\pm(t_1, x_1), \dots, \Psi_0^\pm(t_n, x_n) S)). \quad (22)$$

Here the symbol 0 near Ψ^\pm means that Ψ_0^\pm are operators in the Dirac representation (representation of interaction). The S -matrix has the form

$$S = T \exp(-i \int_{-\infty}^{+\infty} V(t) dt), \quad (23)$$

and

$$S^{-1} = \tilde{T} \exp(i \int_{-\infty}^{+\infty} V(t) dt). \quad (24)$$

Here \tilde{T} is a symbol of antichronological ordering.

Let us recall the basics elements of Nonequilibrium diagram technique. The vertices coming from T -exponent are marked by symbol $-$. The vertices coming from \tilde{T} -exponent are marked by symbol $+$. There exists four tips of propagators

$$\begin{aligned} G_0^{+-}(t_1 - t_2, x_1 - x_2) &= \rho_0(\Psi(t_1, x_1)\Psi^+(t_2, x_2)), \\ G_0^{-+}(t_1 - t_2, x_1 - x_2) &= \rho_0(\Psi^+(t_2, x_2)\Psi(t_1, x_1)), \\ G_0^{--}(t_1 - t_2, x_1 - x_2) &= \rho_0(T(\Psi(t_1, x_1)\Psi^+(t_2, x_2))), \\ G_0^{++}(t_1 - t_2, x_1 - x_2) &= \rho_0(\tilde{T}(\Psi(t_1, x_1)\Psi^+(t_2, x_2))). \end{aligned} \quad (25)$$

Let us write the table of propagators

$$\begin{aligned} G_0^{+-}(t, x) &= \int \frac{d^4k}{(2\pi)^4} (2\pi)\delta(\omega - \omega(k))(1 + n(k))e^{-i(\omega t - kx)}, \\ G_0^{-+}(t, x) &= \int \frac{d^4k}{(2\pi)^4} (2\pi)\delta(\omega - \omega(k))n(k)e^{-i(\omega t - kx)}, \\ G_0^{--}(t, x) &= i \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{1 + n(k)}{\omega - \omega(k) + i0} - \frac{n(k)}{\omega - \omega(k) - i0} \right\} e^{-i(\omega t - kx)}, \\ G_0^{++}(t, x) &= i \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{n(k)}{\omega - \omega(k) + i0} - \frac{1 + n(k)}{\omega - \omega(k) - i0} \right\} e^{-i(\omega t - kx)}. \end{aligned} \quad (26)$$

5 Divergences

A typical example of divergent diagram is pictured on fig1.

fig. 1



The ovals represent the sum of one-particle irreducible diagrams. These diagrams are called chain diagrams. Let us suppose that all divergences of self-energy parts (ovals) are subtracted. The divergences arises from the fact that singular supports of propagators are coincide. At first we consider diagrams

with one self-energy insertion (one-chain diagram). These diagrams are pictured on fig. 2.

fig. 2



These diagrams are analogues to one-loop diagrams in quantum field theory.

There exist two possible types of counterterms. The first one is a counterterms of mass renormalization. Mass renormalization is equivalent to the following replacement

$$\lambda V \rightarrow \lambda V + M, \quad (27)$$

where M has the form

$$M = \int m(k) a^+(k) a(k) d^3 k, \quad (28)$$

$m(k)$ is a real-valued function of k .

The second type of counterterms are counterterms of asymptotical state. Asymptotical state renormalization is equal to the following replacement

$$\rho_0(\cdot) \rightarrow \frac{1}{Z} \rho_0(e^{-\int_{-\infty}^{+\infty} h(t) dt}(\cdot)), \quad (29)$$

where

$$h = \int h(k) a^+(k) a(k) d^3 k,$$

$h(k)$ is a real-valued function and

$$Z = \rho_0(e^{-\int_{-\infty}^{+\infty} h(t) dt}). \quad (30)$$

We will proof below that the counterterms of asymptotical state are enough for the renormalization of all one- and two-chain diagrams.

6 Proof of the existence of divergences in the theory

Suppose that there no divergences in Keldysh diagram technique if $n(k) \neq \frac{1}{e^{\alpha \frac{k^2}{2} + \beta} + 1}$ for any positive α, β . Therefore the Green function

$$\rho(S^{-1}(S\Psi^+(t_1, x_1)\Psi(t_2, x_2))) \quad (31)$$

is translation invariant. So the density matrix

$$\rho_t(x_1, x_2) := \rho(S^{-1}(S\Psi^+(t, x_1)\Psi(t, x_2))) \quad (32)$$

is an integral of motion. Let

$$\rho_t(k) = \int d^3x \rho_t(0, x) e^{ikx}. \quad (33)$$

In zero order of perturbation theory $\rho(k) = n(k)$. But if there no divergences in Keldysh diagram technique it is possible (see [3]) to derive the following kinetic equation for $\rho(k)$

$$\begin{aligned} & \frac{\partial \rho_t(k)}{\partial t} \\ &= \int w(p, p_1 | p_2, p_3) \{ (1 + \rho(p))(1 + \rho(p_1))\rho(p_2)\rho(p_3) \\ & \quad - \rho(p)\rho(p_1)(1 + \rho(p_2))(1 + \rho(p_3)) \}. \end{aligned} \quad (34)$$

The right hand side of this equation is equal to zero only if

$$\rho(k) = \frac{1}{e^{\alpha \frac{k^2}{2} + \beta} + 1} \quad (35)$$

for some α, β ($\alpha > 0, \beta > 0$). But $n(k) = \rho(k)$ in zero order of perturbation theory, so $n(k)$ has a Bose-Einstein form. This contradiction proves our statement.

7 Regularization

Let us now introduce regularization. Note that

$$\frac{1}{x + i\varepsilon} = \frac{x}{x^2 + \varepsilon} - \pi \frac{i}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}. \quad (36)$$

Therefore we use the following regularization

$$\begin{aligned}\delta(\omega - \omega(k)) &\rightarrow \frac{1}{\pi} \frac{\varepsilon}{(\omega - \omega(k))^2 + \varepsilon^2} =: \delta_\varepsilon(\omega - \omega(k)), \\ \mathcal{P}\left(\frac{1}{\omega - \omega(k)}\right) &\rightarrow \frac{\omega - \omega(k)}{(\omega - \omega(k))^2 + \varepsilon^2} =: \mathcal{P}_\varepsilon\left(\frac{1}{\omega - \omega(k)}\right).\end{aligned}\quad (37)$$

8 Some simple relation on the Green functions

Lemma 1. The following equalities hold

$$G^{--}(t_1 - t_2, x_1 - x_2)^* = G^{++}(t_2 - t_1, x_2 - x_1), \quad (38)$$

$$G^{+-}(t_1 - t_2, x_1 - x_2)^* = G^{+-}(t_2 - t_1, x_2 - x_1). \quad (39)$$

Proof. We have

$$\begin{aligned}G^{--}(t_1 - t_2, x_1 - x_2)^* &= \rho_0(T(\Psi(t_1, x_1)\Psi^+(t_2, x_2)))^* \\ &= \rho_0(\tilde{T}(\Psi^+(t_1, x_1)\Psi(t_2, x_2))) = G^{++}(t_2 - t_1, x_2 - x_1).\end{aligned}\quad (40)$$

So the equality 38 is proved. We have

$$\begin{aligned}G^{+-}(t_1 - t_2, x_1 - x_2)^* &= \rho_0(\Psi(t_1, x_1)\Psi^+(t_2, x_2))^* \\ &= \rho_0(\Psi(t_2, x_2)\Psi^+(t_1, x_1)) = G^{+-}(t_2 - t_1, x_2 - x_1).\end{aligned}\quad (41)$$

So the equality 39 is proved.

The Lemma is proved.

It is easy to prove the following

Lemma 2. The following equality holds

$$\begin{aligned}G^{+-}(t, x) &= \theta(t)G^{--}(t, x) + \theta(-t)G^{++}(t, x), \\ G^{-+}(t, x) &= \theta(t)G^{++}(t, x) + \theta(-t)G^{--}(t, x)\end{aligned}\quad (42)$$

Let us introduce the following matrix

$$G = \begin{pmatrix} G^{++} & G^{+-} \\ G^{-+} & G^{--} \end{pmatrix}. \quad (43)$$

Let us introduce the similar matrix for the self-energy operator

$$\Sigma = \begin{vmatrix} \Sigma^{++} & \Sigma^{+-} \\ \Sigma^{-+} & \Sigma^{--} \end{vmatrix}. \quad (44)$$

Dyson equations in Fourier representation has the form

$$G = G_0 + G_0 \Sigma G. \quad (45)$$

We have from these equations that

$$\Sigma = G_0^{-1} - C^{-1}, \quad (46)$$

or in the matrix form

$$\Sigma = \frac{1}{\det G_0} \begin{vmatrix} G_0^{--} & -G_0^{+-} \\ -G_0^{-+} & G_0^{++} \end{vmatrix} - \frac{1}{\det G} \begin{vmatrix} G^{--} & -G^{+-} \\ -G^{-+} & G^{++} \end{vmatrix}. \quad (47)$$

It follows from Lemma 1 that

$$\begin{aligned} G^{++}(\omega, p) &= G^{--}(\omega, p)^* \\ G^{+-}(\omega, p) &= G^{+-}(\omega, p)^*, \\ G^{-+}(\omega, p) &= G^{-+}(\omega, p)^*. \end{aligned} \quad (48)$$

Therefore $\det G_0$, $\det G$ are real and we have the following

Lemma 3.

$$\Sigma^{--}(t_1 - t_2, x_1 - x_2)^* = \Sigma^{++}(t_2 - t_1, x_2 - x_1), \quad (49)$$

$$\Sigma^{+-}(t_1 - t_2, x_1 - x_2)^* = \Sigma^{+-}(t_2 - t_1, x_2 - x_1). \quad (50)$$

The following Lemma holds.

Lemma 4.

$$\Sigma^{++}(\omega, p) + \Sigma^{--}(\omega, p) = -\Sigma^{-+}(\omega, p) - \Sigma^{+-}(\omega, p). \quad (51)$$

Proof. The statement of lemma follows from the Dyson equation (47) and the following two obvious equalities:

$$\begin{aligned} G^{++}(\omega, p) + G^{--}(\omega, p) &= G^{-+}(\omega, p) + G^{+-}(\omega, p), \\ G_0^{++}(\omega, p) + G_0^{--}(\omega, p) &= G_0^{-+}(\omega, p) + G_0^{+-}(\omega, p). \end{aligned} \quad (52)$$

9 Calculation of the propagators in one-chain approximation

Lemma 5. The following limit equality holds (in the sense of distributions):

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} (\delta_\varepsilon^2(x) - \frac{1}{2\pi\varepsilon} \delta_\varepsilon(x)) &= 0, \\
\lim_{\varepsilon \rightarrow 0} (\frac{1}{\varepsilon} \delta_\varepsilon(x) - \frac{1}{\varepsilon} \delta(x)) &= \text{reg}, \\
\lim_{\varepsilon \rightarrow 0} \{ \frac{1}{\pi} \frac{1}{x^2 + \varepsilon^2} - \frac{1}{\varepsilon} \delta(x) \} &= \text{reg}, \\
\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon(x) \mathcal{P}_\varepsilon(\frac{1}{x}) \text{reg} &=, \\
&x \in \mathbb{R}.
\end{aligned} \tag{53}$$

Here reg means some correct distribution.

Proof. Let $f(x)$ be some test function with compact support. We have

$$\begin{aligned}
\int \delta_\varepsilon^2(x) f(x) &= \frac{1}{\pi^2} \int \frac{1}{(x^2 + \varepsilon^2)^2} f(x) dx \\
&= \frac{1}{\pi^2} \int \frac{\varepsilon^2}{(x^2 + \varepsilon^2)^2} \{f(0) + x f'(0) + x^2 \psi(x)\} dx
\end{aligned} \tag{54}$$

for some smooth bounded function $\psi(x)$. We have

$$\begin{aligned}
\int \delta_\varepsilon^2(x) f(x) &= \frac{1}{\varepsilon \pi^2} \int \frac{1}{(x^2 + 1)^2} \{f(0) + \varepsilon x f'(0) + \varepsilon^2 x^2 \psi(\varepsilon x)\} \\
&= \frac{1}{\pi^2} \{ \frac{1}{\varepsilon} \int \frac{1}{(x^2 + 1)^2} dx \} f(0) + O(\varepsilon).
\end{aligned} \tag{55}$$

But

$$\int \frac{1}{(x^2 + 1)^2} dx = \frac{\pi}{2}. \tag{56}$$

So

$$\int \delta_\varepsilon^2(x) f(x) = \frac{1}{2\pi\varepsilon} f(0) + O(\varepsilon). \tag{57}$$

So the first equality is proved. One can prove other three equality by the same way.

Therefore we see from the Lemmas 1,2, that we can only consider the function $G^{--}(t, x)$. But the function $G^{--}(t, x)$ can be represented as a sum of chain diagrams. At first let us consider the diagrams with one self-energy insertion (one-chain diagram). We have $G_\varepsilon^{--} = \sum_{i,j=\pm} H_\varepsilon^{ij}$, where the diagrams for H_ε^{ij} are presented at the fig. 2. We have the following representation for the divergent parts of these diagrams.

$$\begin{aligned}
& (H_\varepsilon^{--})_{div}(\omega, p) + (H_\varepsilon^{++})_{div}(\omega, p) \\
&= 2\pi \Sigma^{--}(\omega, p) n(p) (1 + n(p)) \frac{1}{\varepsilon} \delta(\omega - \omega(p)) \\
&+ 2\pi \Sigma^{++}(\omega, p) n(p) (1 + n(p)) \frac{1}{\varepsilon} \delta(\omega - \omega(p)). \tag{58}
\end{aligned}$$

We see that the divergent part of these two diagrams is real (because $\Sigma^{--} = (\Sigma^{++})^*$).

It is obvious from previous calculations that the sum of two possible mass-renormalization diagrams is equal to zero. Let us consider the singular part of other two diagram presented at fig 3.

fig. 3



We have

$$\begin{aligned}
& (H_\varepsilon^{-+})_{div}(\omega, p) + (H_\varepsilon^{+-})_{div}(\omega, p) \\
&= \pi(2\pi)(1 + 2n(p))(1 + n(p))\delta_\varepsilon^2(\omega - \omega(p))\Sigma^{-+}(\omega, p) \\
&+ \pi(2\pi)(1 + 2n(p))n(p)\delta_\varepsilon^2(\omega - \omega(p))\Sigma^{+-}(\omega, p) \\
&= \pi(1 + 2n(p))(1 + n(p))\frac{1}{\varepsilon}\delta(\omega - \omega(p))\Sigma^{-+}(\omega, p) \\
&+ \pi(1 + 2n(p))n(p)\frac{1}{\varepsilon}\delta(\omega - \omega(p))\Sigma^{+-}(\omega, p) + O(\varepsilon). \tag{59}
\end{aligned}$$

We see that $(H_\varepsilon^{--})_{div}(\omega, p) + (H_\varepsilon^{++})_{div}(\omega, p)$, $(H_\varepsilon^{-+})_{div}(\omega, p)$ and $(H_\varepsilon^{+-})_{div}(\omega, p)$ are real.

We will use the dotted line for lines which connects creation-annihilation operators with operators coming from the vertex: $\int h(k)a^+(k)a(k)d^3k$ (see

fig. 4



fig. 4).

So the divergences in $G_\epsilon^{--} = \sum_{i,j=\pm} H_\epsilon^{ij}$ can be subtracted by the following counterterm:

$$h(p) = \Sigma^{++}(\omega, p) + \Sigma^{--}(\omega, p) + \frac{(1 + 2n(p))}{2n(p)(1 + n(p))} \times \{(1 + n(p))\Sigma^{-+}(\omega, p) + n(p)\Sigma^{+-}(\omega, p)\}. \quad (60)$$

We have by using Lemma 4

$$h(p) = \frac{1 + 2n(p)}{2(1 + n(p))n(p)} \times \{(1 + n(p))\Sigma^{-+}(\omega, p) - n(p)\Sigma^{+-}(\omega, p)\}. \quad (61)$$

The left hand side of this equation can be rewritten as follows (in approximation used in [3])

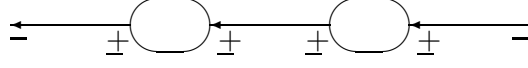
$$h(p) = \frac{1 + 2n(p)}{2n(p)(1 + n(p))} St(p), \quad (62)$$

where $St(p)$ is a scattering integral. So $h(p) \neq 0$ for non-equilibrium matter.

10 Calculation of propagators in two-chain approximation

We will calculate the divergent parts of all diagrams H_ϵ^{ijkl} , $i, j, k, l = \pm$ presented at fig. 5.

fig. 5



We have

$$G_{\varepsilon}^{--} = \sum_{i,j=\pm} H^{ij} + \sum_{i,j,k,l=\pm} H_{\varepsilon}^{ijkl} \text{ in two-chain approximation.}$$

Let us calculate $H_{\varepsilon}^{----}(\omega, p)$ (fig. 6). We have

$$\begin{aligned} H_{\varepsilon}^{----}(\omega, p) &= -(\Sigma^{--})^2 \times i \left\{ \frac{1+n(p)}{\omega - \omega(p) + i\varepsilon} - \frac{n(p)}{\omega - \omega(p) - i\varepsilon} \right\}^3 \\ &= -i(\Sigma^{--})^2 \left\{ \frac{(1+n(p))^3}{(\omega - \omega(p) + i\varepsilon)^3} - \frac{(n(p))^3}{(\omega - \omega(p) - i\varepsilon)^3} \right. \\ &\quad \left. + \frac{3(1+n(p))n(p)^2}{(\omega - \omega(p) + i\varepsilon)(\omega - \omega(p) - i\varepsilon)^2} - \frac{3(1+n(p))^2n(p)}{(\omega - \omega(p) + i\varepsilon)^2(\omega - \omega(p) - i\varepsilon)} \right\}. \end{aligned} \quad (63)$$

But $\left(\frac{1}{\omega - \omega(p) \pm i\varepsilon} \right)^n$ is a distribution. So we have the following expression for the singular part of H_{ε}^{----} .

$$\begin{aligned} H_{\varepsilon}^{----}(\omega, p) &= -\frac{3in(p)^2(1+n(p))}{(\omega - \omega(p) + i\varepsilon)(\omega - \omega(p) - i\varepsilon)^2} \\ &\quad + \frac{3in(k)(1+n(k))^2}{(\omega - \omega(p) + i\varepsilon)^2(\omega - \omega(p) - i\varepsilon)} + O(1). \end{aligned} \quad (64)$$

We have

$$\begin{aligned} &\frac{1}{(\omega - \omega(p) + i\varepsilon)^2(\omega - \omega(p) - i\varepsilon)} \\ &= \frac{1}{(\omega - \omega(p) + i\varepsilon)^2} \left\{ \frac{1}{(\omega - \omega(p) + i\varepsilon)} + 2\pi i \delta_{\varepsilon}(\omega) \right\} \\ &= \frac{2\pi i}{(\omega - \omega(p) + i\varepsilon)^2} \delta_{\varepsilon}(\omega) + O(1). \end{aligned} \quad (65)$$

Let $f(\omega)$ be a test function $f(\omega) \in S$. We will calculate

$$I_{\varepsilon} := \frac{1}{\pi \varepsilon^2} \int \frac{1}{(\Omega + i)^2} \frac{1}{\Omega^2 + 1} f(\varepsilon \Omega) d\Omega. \quad (66)$$

But $f(\varepsilon\omega) = f(0) + \varepsilon\Omega f'(0) + \dots$. Substituting this decomposition into last equation, we find

$$\begin{aligned} I_\varepsilon &= \frac{1}{\pi\varepsilon^2} f(0) \int \frac{1}{(\Omega + i)^2(\Omega^2 + 1)} d\Omega \\ &+ \frac{1}{\pi\varepsilon} f'(0) \int \frac{\Omega}{(\Omega + i)^2(\Omega^2 + 1)} d\Omega + O(1). \end{aligned} \quad (67)$$

We use the Cauchy theorem for calculation these two integrals. Let us close the integration contour in the upper half-plane. The integrand has only one pole at the upper half-plane at the point $\Omega = i$. Therefore

$$\begin{aligned} A &:= \int_{-\infty}^{+\infty} \frac{1}{(\Omega + i)^2(\Omega^2 + 1)} d\Omega \\ &= 2\pi i \frac{1}{(\Omega + i)^3} \Big|_{\Omega=i} = 2\pi i \times \frac{1}{-8i} = -\frac{\pi}{4}. \end{aligned} \quad (68)$$

By the same way we find

$$\begin{aligned} B &:= \int \frac{\Omega}{(\Omega + i)(\Omega^2 + 1)} d\Omega \\ &= 2\pi i \frac{\Omega}{(\Omega + i)^3} \Big|_{\Omega=i} = -\frac{\pi i}{4}. \end{aligned} \quad (69)$$

So we have

$$I_\varepsilon = -\frac{1}{\varepsilon^2} \frac{1}{4} f(0) - \frac{i}{4\varepsilon} f'(0) + O(1), \quad (70)$$

or

$$\begin{aligned} &\left(\frac{1}{\omega - \omega(p) + i\varepsilon} \right)^2 \delta_\varepsilon(\omega - \omega(p)) \\ &= -\frac{1}{4\varepsilon^2} f(\omega - \omega(p)) + \frac{i}{4\varepsilon} \delta'(\omega - \omega(p)) \\ &\quad + O(1). \end{aligned} \quad (71)$$

In result

$$\frac{1}{(\omega - \omega(p) + i\varepsilon)^2(\omega - \omega(p) - i\varepsilon)}$$

$$\begin{aligned}
&= 2\pi i \left(-\frac{1}{4\varepsilon^2} \delta(\omega - \omega(p)) + \frac{i}{4\varepsilon} \delta'(\omega - \omega(p)) + O(1) \right) \\
&\quad \frac{1}{(\omega - \omega(p) + i\varepsilon)(\omega - \omega(p) - i\varepsilon)^2} \\
&= -2\pi i \left(-\frac{1}{4\varepsilon^2} \delta(\omega - \omega(p)) - \frac{i}{4\varepsilon} \delta'(\omega - \omega(p)) + O(1) \right). \tag{72}
\end{aligned}$$

At last

$$\begin{aligned}
(H_\varepsilon^{----})_{div}(\omega, p) &= \left\{ \frac{3\pi}{2\varepsilon^2} n(p)(1 + n(p))(1 + 2n(p)) \delta(\omega - \omega(p)) \right. \\
&\quad \left. - \frac{6i\pi}{4\varepsilon} n(p)(1 + n(p)) \delta'(\omega - \omega(p)) \right\} (\Sigma^{--}(\omega, p))^2. \tag{73}
\end{aligned}$$

Let us now calculate the diagrams presented at fig. 6,7.

fig. 6

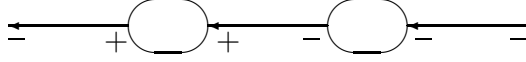
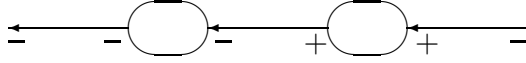


fig. 7



way as previous we find

By the same

$$\begin{aligned}
&(H_\varepsilon^{++--})_{div}(\omega, p) = (H_\varepsilon^{--++})_{div}(\omega, p) \\
&= \frac{2\pi}{4} n(p)(1 + n(p)) \left\{ \frac{3}{\varepsilon^2} (1 + 2n(p)) \delta(\omega - \omega(p)) \right. \\
&\quad \left. - \frac{i}{\varepsilon} \delta'(\omega - \omega(p)) \right\} \Sigma^{--}(\omega, p) \Sigma^{++}(\omega, p). \tag{74}
\end{aligned}$$

Let us now consider diagrams presented at fig. 8, 9.

fig. 8

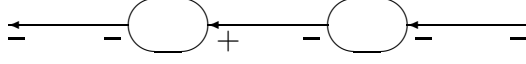
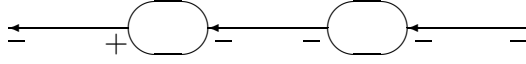


fig. 9



It is easy to

see that

$$H_{\varepsilon}^{-+--}(\omega, p) = \frac{(1+n(p))}{n(p)} \Sigma^{-+}(\omega, p) (\Sigma^{+-}(\omega, p))^{-1} H_{\varepsilon}^{--+-}(\omega, p). \quad (75)$$

Omitting the calculations we find

$$\begin{aligned} (H_{\varepsilon}^{-+--})_{div}(\omega, p) &= (H_{\varepsilon}^{-+--})_{div}(\omega, p) \\ &= 2\pi(1+n(p))\Sigma^{-+}(\omega, p)\Sigma^{--}(\omega, p) \\ &\times \left\{ \left[\frac{1}{4}((1+n(p))^2 + n(p)^2) + n(p)(1+n(p)) \right] \frac{\delta(\omega - \omega(p))}{\varepsilon^2} \right. \\ &\quad \left. + \frac{-i}{4\varepsilon}(1+2n(p))\delta'(\omega - \omega(p)) \right\} \end{aligned} \quad (76)$$

and

$$\begin{aligned} (H_{\varepsilon}^{--+-})_{div}(\omega, p) &= (H_{\varepsilon}^{--+-})_{div}(\omega, p) \\ &= 2\pi n(p)\Sigma^{+-}(\omega, p)\Sigma^{--}(\omega, p) \\ &\times \left\{ \left[\frac{1}{4}((1+n(p))^2 + n(p)^2) + n(p)(1+n(p)) \right] \frac{\delta(\omega - \omega(p))}{\varepsilon^2} \right. \\ &\quad \left. + \frac{(-i)}{4\varepsilon}(1+2n(p))\delta'(\omega - \omega(p)) \right\}. \end{aligned} \quad (77)$$

Let us now present analytical expression for other diagrams:

$$\begin{aligned} (H_{\varepsilon}^{++++})_{div}(\omega, p) &= \left\{ (1+n(p))n(p)(1+2n(p)) \frac{3\pi}{2\varepsilon^2} \delta(\omega - \omega(p)) \right. \\ &\quad \left. + \frac{\pi i}{2\varepsilon} \delta'(\omega - \omega(p))(1+n(p))n(p) \right\} (\Sigma^{++}(\omega, p))^2, \end{aligned} \quad (78)$$

$$(H_\varepsilon^{++++})_{div}(\omega, p) = \Sigma^{++}(\omega, p)\Sigma^{+-}(\omega, p)\frac{\pi}{\varepsilon^2} \\ \times (1 + 3n(p) + 3n(p)^2)n(p)\delta(\omega - \omega(p)), \quad (79)$$

$$(H_\varepsilon^{-+++})_{div}(\omega, p) = \Sigma^{-+}(\omega, p)\Sigma^{++}(\omega, p) \\ \times (1 + n(p))(1 + 3n(p) + 3n(p)^2)\frac{\pi}{\varepsilon^2}\delta(\omega - \omega(p)), \quad (80)$$

$$(H_\varepsilon^{+-++})_{div}(\omega, p) = 3\pi\Sigma^{+-}(\omega, p)\Sigma^{++}(\omega, p)\delta(\omega - \omega(p)) \\ \times \frac{(1 + n(p))n(p)^2}{\varepsilon^2}, \quad (81)$$

$$(H_\varepsilon^{++-+})_{div}(\omega, p) = 3\pi\Sigma^{++}(\omega, p)\Sigma^{-+}(\omega, p)\delta(\omega - \omega(p)) \\ \times \frac{(1 + n(p))^2n(p)}{\varepsilon^2}, \quad (82)$$

$$(H_\varepsilon^{+--+})_{div}(\omega, p) \\ = \left(\frac{2\pi}{4}\right)n(p)(1 + n(p))\Sigma^{+-}(\omega, p)\Sigma^{-+}(\omega, p) \\ \times n(p)(1 + n(p))\left\{-\frac{3}{\varepsilon^2}(1 + 2n(p))\delta(\omega - \omega(p)) - \frac{i}{\varepsilon}\delta'(\omega - \omega(p))\right\}, \quad (83)$$

$$(H_\varepsilon^{-++-})_{div}(\omega, p) = \frac{2\pi}{4\varepsilon^2}\{n(p)^3 + 2n(p)(1 + n(p))^2 \\ + 2n(p)^2(1 + n(p)) + (1 + n(p))^3\} \\ - \frac{2\pi}{4\varepsilon}i\delta'(\omega - \omega(p)) \\ \times \Sigma^{-+}(\omega, p)\Sigma^{+-}(\omega, p)\{1 + n(p) + n(p)^2\}, \quad (84)$$

$$(H_\varepsilon^{-+-+})_{div}(\omega, p) = -\frac{2\pi}{4}(1 + n(p))^2\Sigma^{-+}(\omega, p)\Sigma^{-+}(\omega, p) \\ \times \left\{-\frac{3}{\varepsilon^2}(1 + 2n(p))\delta(\omega - \omega(p)) + \frac{i}{\varepsilon}\delta'(\omega - \omega(p))\right\}, \quad (85)$$

$$(H_\varepsilon^{+--+})_{div}(\omega, p) = -\frac{2\pi}{4}(n(p))^2\Sigma^{+-}(\omega, p)\Sigma^{+-}(\omega, p) \\ \times \left\{-\frac{3}{\varepsilon^2}(1 + 2n(p))\delta(\omega - \omega(p)) + \frac{i}{\varepsilon}\delta'(\omega - \omega(p))\right\}. \quad (86)$$

11 Counterterm diagrams

Let us recall that we renormalize the asymptotical state ρ by the following way

$$\rho(\cdot) \rightarrow \rho(e^{-\int_{-\infty}^{+\infty} h(t) dt}(\cdot)) \frac{1}{Z}, \quad (87)$$

where

$$Z = \rho(e^{-\int_{-\infty}^{+\infty} h(t) dt}), \quad (88)$$

and

$$h(t) = e^{itH_0} h e^{-itH_0}. \quad (89)$$

So we have to take into account the counterterm diagrams pictured at fig. 10, 11. i.e. $H^{00\pm\pm}$ and $H^{\pm\pm 00}$ respectively.

fig. 10

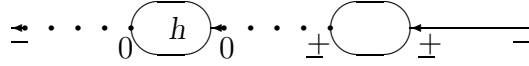
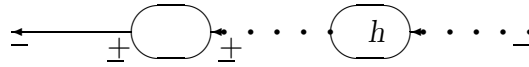


fig. 11



late H^{00--} . We have

Let us calcu-

$$\begin{aligned} (H_\varepsilon^{00--})_{div} &= -h(\omega(p), p)(2\pi)^2 \delta_\varepsilon^2(\omega - \omega(p)) n(p)(1 + n(p)) \\ &\quad \times i\Sigma^{--}(\omega, p) \left\{ \frac{1 + n(p)}{\omega - \omega(p) + i\varepsilon} - \frac{n(p)}{\omega - \omega(p) - i\varepsilon} \right\} \\ &= -i(2\pi)^2 h(p, \omega(p)) \Sigma^{--}(\omega, p) n(p)(1 + n(p)) \times \end{aligned}$$

$$\begin{aligned} & \{ [\frac{1}{\varepsilon^2}(\frac{-3i}{8\pi})\delta(\omega - \omega(p)) - \frac{1}{\varepsilon}(\frac{1}{8\pi})\delta'(\omega - \omega(p))] (1 + n(p)) \\ & - n(p) [\frac{1}{\varepsilon^2}(\frac{3i}{8\pi})\delta(\omega - \omega(p)) - \frac{1}{\varepsilon}\frac{1}{8\pi}\delta'(\omega - \omega(p))] \}. \end{aligned} \quad (90)$$

In result

$$\begin{aligned} (H_\varepsilon^{00--})_{div}(\omega, p) &= -h(\omega(p), p)\Sigma^{--}(\omega, p)n(p)(1 + n(p))(2\pi)^2 \\ & \times \{ \frac{1}{\varepsilon^2}\frac{3}{8\pi}(1 + 2n(p))\delta(\omega - \omega(p)) - \frac{1}{\varepsilon}\frac{i}{8\pi}\delta'(\omega - \omega(p)) \}. \end{aligned} \quad (91)$$

Let us present now the analytical expression for G^{00+-} . Omitting some calculation we have:

$$\begin{aligned} (H_\varepsilon^{00+-})_{div}(\omega, p) &= -h(\omega(p), p)(2\pi)^2\Sigma^{+-}(p, \omega)n(p)(1 + n(p)) \\ & \times \{ \frac{1}{\varepsilon^2}\delta(\omega - \omega(p))\frac{3}{8\pi}(1 + 2n(p)) - \frac{i}{\varepsilon}\frac{1}{8\pi}\delta'(\omega - \omega(p)) \}. \end{aligned} \quad (92)$$

Let us present now without calculations all other counterterm diagrams:

$$\begin{aligned} (H_\varepsilon^{00-+})_{div}(\omega, p) &= -h(\omega(p), p)\Sigma^{-+}(\omega, p)\frac{3\pi}{\varepsilon^2}\delta(\omega - \omega(p)) \\ & \times n(p)(1 + n(p))^2, \end{aligned} \quad (93)$$

$$\begin{aligned} (H_\varepsilon^{00++})_{div}(\omega, p) &= -\frac{3\pi}{\varepsilon^2}h(\omega(p), p)\Sigma^{++}(\omega, p) \\ & \times n(p)(1 + n(p))^2\delta(\omega - \omega(p)), \end{aligned} \quad (94)$$

$$\begin{aligned} (H_\varepsilon^{-00})_{div}(\omega, p) &= -h(\omega(p), p)\Sigma^{--}(\omega, p)n(p)(1 + n(p)) \\ & \times (2\pi)^2 \{ \frac{1}{\varepsilon^2}\delta(\omega - \omega(p))\frac{3}{8\pi}(1 + 2n(p)) - \frac{i}{\varepsilon}\frac{1}{8\pi}\delta'(\omega - \omega(p)) \}, \end{aligned} \quad (95)$$

$$\begin{aligned} (H_\varepsilon^{-+00})_{div}(\omega, p) &= -h(\omega(p), p)\Sigma^{-+}(\omega, p)n(p)(1 + n(p)) \\ & \times (2\pi)^2 \{ \frac{1}{\varepsilon^2}\delta(\omega - \omega(p))\frac{3}{8\pi}(1 + 2n(p)) - \frac{i}{\varepsilon}\frac{1}{8\pi}\delta'(\omega - \omega(p)) \}, \end{aligned} \quad (96)$$

$$\begin{aligned} (H_\varepsilon^{+\pm 00})_{div}(\omega, p) &= -\frac{3\pi}{\varepsilon^2}n(p)^2(1 + n(p))\Sigma^{+\pm}h(\omega(p), p) \\ & \times \delta(\omega - \omega(p)). \end{aligned} \quad (97)$$

We have presented all counterterm diagrams.

12 Annihilation of all strong divergences in the Green functions

Let $f(\varepsilon)$ be a function of ε of the form

$$f(\varepsilon) = c_2 \frac{1}{\varepsilon^2} + c_1 \frac{1}{\varepsilon} + O(1), \text{ as } \varepsilon \rightarrow 0. \quad (98)$$

Put by definition

$$\begin{aligned} (f(\varepsilon))_2 &= c_2 \frac{1}{\varepsilon^2}, \\ (f(\varepsilon))_1 &= c_1 \frac{1}{\varepsilon}. \end{aligned} \quad (99)$$

We find that $(H_\varepsilon^{--++})_2, (H_\varepsilon^{++--})_2, (H_\varepsilon^{-++-})_2, (H_\varepsilon^{+--+})_2, (H_\varepsilon^{--++})_2, (H_\varepsilon^{++--})_2$ are real. But the following terms $(H_\varepsilon^{----})_2$ and $(H_\varepsilon^{++++})_2$ are complex-conjugated to each other.

All not real counterterm diagrams have the form:

$$\begin{aligned} (H_\varepsilon^{00--}(\omega, p))_2 &= -(2\pi)^2 h(\omega(p), p) n(p) (1 + n(p)) (1 + 2n(p)) \\ &\quad \times \frac{3}{8\pi} \frac{1}{\varepsilon^2} \Sigma^{--}(\omega, p) \delta(\omega - \omega(p)), \\ (H_\varepsilon^{00++}(\omega, p))_2 &= -3\pi h(\omega(p), p) n(p) (1 + n(p))^2 \\ &\quad \times \Sigma^{++}(\omega, p) \delta(\omega - \omega(p)), \\ (H_\varepsilon^{--00}(\omega, p))_2 &= -h(\omega(p), p) n(p) (1 + n(p)) (1 + 2n(p)) \frac{3}{8\pi} (2\pi)^2 \\ &\quad \times \Sigma^{--}(\omega, p) \delta(\omega - \omega(p)), \\ (H_\varepsilon^{++00}(\omega, p))_2 &= -h(\omega(p), p) n(p)^2 (1 + n(p)) \frac{3\pi}{\varepsilon^2} \\ &\quad \times \Sigma^{++}(\omega, p) \delta(\omega - \omega(p)). \end{aligned} \quad (100)$$

So it is easy to see that the sum of counterterm diagram is real.

One can see that the sum

$$(H_\varepsilon^{++++})_2 + (H_\varepsilon^{--++})_2 + (H_\varepsilon^{++--})_2 + (H_\varepsilon^{----})_2 \quad (101)$$

is real and

$$(H_\varepsilon^{+++-})_2 + (H_\varepsilon^{+-++})_2 + (H_\varepsilon^{+---})_2 + (H_\varepsilon^{--+-})_2 \quad (102)$$

is real too. So all the most strong divergences can be renormalized by renormalization of the asymptotical state.

13 Annihilation of all weak divergences in the Green functions

Now we try to answer the question: if the divergences which are proportional to $\delta'(\omega - \omega(p))$ in Green functions are vanished.

Let us recall what

$$\Sigma^{-+} + \Sigma^{+-} = -\Sigma^{++} - \Sigma^{--}. \quad (103)$$

Here and below we omit arguments (ω, p) of all functions. We have

$$(H_{\varepsilon}^{----})_1 = -(\Sigma^{--})^2 \frac{6i\pi}{4\varepsilon} n(1+n)\delta'. \quad (104)$$

Here and below we will omit an argument of δ -function. Corresponding counterterm is equal

$$(H_{\varepsilon C}^{----})_1 = (\Sigma^{--})^2 \frac{i\pi}{\varepsilon} n(1+n)\delta'. \quad (105)$$

Therefore

$$(H_{\varepsilon R}^{----})_1 = -(\Sigma^{--})^2 \frac{i\pi}{2\varepsilon} n(1+n)\delta'. \quad (106)$$

Here we put by definition

$$H_{\varepsilon R}^{\pm\pm\pm\pm} = H_{\varepsilon}^{\pm\pm\pm\pm} + H_{\varepsilon C}^{\pm\pm\pm\pm}. \quad (107)$$

We have also

$$(H_{\varepsilon}^{++++})_1 = (\Sigma^{++})^2 \frac{i\pi}{2\varepsilon} n(1+n)\delta', \quad (108)$$

and corresponding counterterms are equal to zero. In result

$$(H_{\varepsilon R}^{----})_1 + (H_{\varepsilon R}^{++++})_1 = -\frac{i\pi}{2\varepsilon} \times \{(\Sigma^{--})^2 - (\Sigma^{++})^2\} n(1+n)\delta'.$$

This quantity is real.

Now let us find counterterms to the diagrams

$$H_{\varepsilon}^{++--} \text{ and } H_{\varepsilon}^{--++}. \quad (109)$$

By using the table of divergent parts of diagrams we find:

$$(H_{\varepsilon C}^{++--})_1 = +\frac{i\pi}{2\varepsilon}\delta'n(1+n)\Sigma^{--}\Sigma^{++}. \quad (110)$$

By the same way we find

$$(H_{\varepsilon C}^{--++})_1 = +\frac{i\pi}{2\varepsilon}\delta'n(1+n)\Sigma^{--}\Sigma^{++}, \quad (111)$$

so

$$(H_{\varepsilon R}^{++--} + H_{\varepsilon R}^{--++})_1 = 0. \quad (112)$$

Now we will calculate the diagram H_{ε}^{-++} .

$$(H_{\varepsilon}^{-++})_1 = -\frac{2\pi}{4\varepsilon\varepsilon}(1+n)^2\frac{i}{\varepsilon}(\Sigma^{--})^2\delta'. \quad (113)$$

It is easy to find that its counterterm is equal to

$$(H_{\varepsilon C}^{-++})_1 = \frac{\pi}{4}\frac{i}{\varepsilon}(1+2n)(1+n)(\Sigma^{--})^2\delta'. \quad (114)$$

Therefore

$$(H_{\varepsilon}^{-++})_1 = -\frac{\pi i}{4\varepsilon}(1+n)(\Sigma^{--})^2\delta'. \quad (115)$$

Now let us calculate the diagram H^{+-+} .

$$(H_{\varepsilon}^{+-+})_1 = -\frac{2\pi i}{4\varepsilon}n^2(\Sigma^{+-})^2\delta' \quad (116)$$

and

$$(H_{\varepsilon C}^{+-+})_1 = \frac{\pi i}{4\varepsilon}(1+2n)n(\Sigma^{+-})^2\delta'. \quad (117)$$

In result:

$$(H_{\varepsilon R}^{+-+})_1 = \frac{\pi i}{4\varepsilon}n(\Sigma^{+-})^2\delta'. \quad (118)$$

Now let us consider the diagram:

$$(H^{-+--})_1 = \frac{-2\pi i}{4\varepsilon}(1+n(p))(1+2n(p))\Sigma^{--}\Sigma^{--}\delta'. \quad (119)$$

Let us find counterterms to this diagram:

$$\begin{aligned}
(H_{\varepsilon C}^{-+--})_1 &= \frac{\pi i}{2\varepsilon} \Sigma^{-+} \Sigma^{--} (1+n) n \delta' \\
&+ (1+n)(1+2n) \frac{\pi i}{4\varepsilon} \Sigma^{-+} \Sigma^{--} \delta' \\
&= -\frac{\pi i}{4\varepsilon} (1+n) \Sigma^{-+} \Sigma^{--} \delta'.
\end{aligned} \tag{120}$$

Therefore

$$(H_{\varepsilon R}^{-+--})_1 = -\Sigma^{-+} \Sigma^{--} \frac{\pi i}{4\varepsilon} (1+n)(3+8n) \delta'. \tag{121}$$

Let us consider the diagram H^{----+} :

$$(H_{\varepsilon}^{----+})_1 = -\frac{2\pi i}{4\varepsilon} (1+n)(1+2n) \Sigma^{-+} \Sigma^{--} \delta'. \tag{122}$$

The counterterm corresponding to this diagram is equal to

$$(H_{\varepsilon C}^{----+})_1 = \frac{i\pi}{4\varepsilon} (1+n)(1+2n) \Sigma^{-+} \Sigma^{--} \delta'. \tag{123}$$

In result

$$(H_{\varepsilon R}^{-+--})_1 + (G_{\varepsilon R}^{----+})_1 = \frac{-i\pi}{2\varepsilon} (1+n)^2 \Sigma^{-+} \Sigma^{--} \delta'. \tag{124}$$

Now let us calculate counterterms to the following diagrams:

$$\begin{aligned}
&(H_{\varepsilon}^{+---})_1 + (H_{\varepsilon}^{--+-})_1 \\
&= \frac{\pi(-i)}{\varepsilon} n(1+2n) \delta'.
\end{aligned} \tag{125}$$

$$\begin{aligned}
(H_{\varepsilon C}^{--+-})_1 &= \pi i \Sigma^{+-} \Sigma^{--} \left\{ \frac{n(1+n)}{2\varepsilon} \right. \\
&\quad \left. + \frac{n(1+2n)}{4\varepsilon} \right\} \delta'.
\end{aligned} \tag{126}$$

$$(H_{\varepsilon C}^{+---})_1 = \pi i \frac{n(1+2n)}{4\varepsilon} \Sigma^{-+} \Sigma^{--} \delta'. \tag{127}$$

Therefore

$$(H_{\varepsilon C}^{--++})_1 + (H_{\varepsilon C}^{+---})_1 = \frac{-i\pi}{2\varepsilon} n^2 \Sigma^{-+} \Sigma^{--} \delta', \quad (128)$$

and

$$(H_{\varepsilon R}^{+---})_1 + (H_{\varepsilon R}^{--++})_1 = \frac{-i\pi}{2\varepsilon} n^2 \Sigma^{+-} \Sigma^{--} \delta'. \quad (129)$$

It is easy to see that:

$$\begin{aligned} (H_{\varepsilon}^{+---})_1 &= (H_{\varepsilon}^{--++})_1 = (H_{\varepsilon}^{++--})_1 = (H_{\varepsilon}^{+++-})_1 = 0, \\ (H_{\varepsilon C}^{+---})_1 &= 0, \\ (H_{\varepsilon C}^{++--})_1 &= \frac{i\pi}{\varepsilon} \Sigma^{++} \Sigma^{+-} n(1+n) \delta'. \end{aligned} \quad (130)$$

So

$$(H_{\varepsilon R}^{+---} + H_{\varepsilon R}^{++--})_1 = \frac{i\pi}{2\varepsilon} \Sigma^{++} \Sigma^{+-} n(1+n) \delta'. \quad (131)$$

It is easy to find that

$$\begin{aligned} (H_{\varepsilon}^{-+++})_1 + (H_{\varepsilon}^{++--})_1 &= 0, \\ (H_{\varepsilon C}^{-+++})_1 &= n(1+n) \frac{i\pi}{2\varepsilon} \Sigma^{-+} \Sigma^{++} \delta', \\ (H_{\varepsilon C}^{++--})_1 &= 0. \end{aligned} \quad (132)$$

Therefore

$$(H_{\varepsilon R}^{-+++} + H_{\varepsilon}^{++--})_1 = n(1+n) \frac{i\pi}{2\varepsilon} \Sigma^{-+} \Sigma^{++} \delta'. \quad (133)$$

Now we must to calculate the following two diagrams:

$$(H_{\varepsilon}^{+---})_1 \text{ and } (H_{\varepsilon}^{--++})_1. \quad (134)$$

Let us start with $(H_{\varepsilon}^{+---})_1$. We have

$$(H_{\varepsilon C}^{+---})_1 = 0. \quad (135)$$

Therefore

$$(H_{\varepsilon R}^{+--+})_1 = -\frac{2i\pi}{4\varepsilon}\Sigma^{-+}\Sigma^{+-}n(p)(1+n(p))\delta'. \quad (136)$$

Let us calculate the diagram H_{ε}^{----} . We have

$$H_{\varepsilon}^{-++-} = -\frac{\pi i}{2\varepsilon}\{1+n+n^2\}\Sigma^{-+}\Sigma^{+-}\delta', \quad (137)$$

$$(H_{\varepsilon C}^{-++-})_1 = \Sigma^{-+}\Sigma^{+-}\frac{\pi i}{4}((1+2n)(1+2n))\delta'. \quad (138)$$

In result

$$(H_{\varepsilon R}^{-++-} + H_{\varepsilon R}^{+--+}) = -\frac{\pi i}{4\varepsilon}\Sigma^{+-}\Sigma^{-+}\delta'. \quad (139)$$

Now we must summarize all these contribution neglecting by real parts. All real parts can be subtracted by counterterms of asymptotical state. We have

$$\begin{aligned} & \sum_{i,j,k,l=\pm} (H_{\varepsilon R}^{ijkl})_1 \\ &= \left\{ -\frac{\pi i}{4}(1+n)(\Sigma^{-+})^2 + \frac{\pi i}{4}n(\Sigma^{+-})^2 \right. \\ & \quad - \frac{i\pi}{2}\Sigma^{-+}\Sigma^{--}(1+n)^2 - \frac{i\pi}{2}\Sigma^{+-}\Sigma^{--}n^2 \\ & \quad + \frac{\pi i}{2\varepsilon}\Sigma^{++}\Sigma^{+-}n(1+n) + \frac{i\pi}{2\varepsilon}n(1+n)\Sigma^{-+}\Sigma^{++} \\ & \quad \left. - \frac{\pi i}{4\varepsilon}\Sigma^{+-}\Sigma^{-+} \right\} \delta'. \end{aligned} \quad (140)$$

Let us unite in the r.h.s. of the last formula 3rd and 6th terms, and 4th and 5th terms. Neglecting by some real terms we find:

$$\begin{aligned} & \sum_{i,j,k,l=\pm} (H_{\varepsilon R}^{i,j,k,l})_1 \\ &= \left\{ -\frac{\pi i}{4\varepsilon}(\Sigma^{-+})^2 + \frac{\pi i}{4\varepsilon}n(\Sigma^{+-})^2 \right. \\ & \quad - \frac{i\pi}{2\varepsilon}\Sigma^{-+}\Sigma^{++}(1+n) + \frac{\pi i}{2\varepsilon}\Sigma^{++}\Sigma^{+-}n \\ & \quad \left. - \frac{\pi i}{4\varepsilon}\Sigma^{+-}\Sigma^{-+} \right\} \delta'. \end{aligned} \quad (141)$$

It follows from the identity

$$\Sigma^{++} + \Sigma^{--} = -\Sigma^{-+} - \Sigma^{+-} \quad (142)$$

that

$$\Sigma^{++} + \Sigma^{-+} = -\Sigma^{--} - \Sigma^{+-}, \quad (143)$$

and

$$\Sigma^{++} + \Sigma^{+-} = -\Sigma^{--} - \Sigma^{-+}. \quad (144)$$

Let us unite in (141) the first term with 3rd term and second term with 4th term. We find

$$\begin{aligned} & \sum_{i,j,k,l=\pm} (H_{\varepsilon R}^{ijkl})_1 \\ &= -\frac{\pi i}{4} \Sigma^{-+} (\Sigma^{++} - \Sigma^{--} - \Sigma^{+-}) (1+n) \delta' \\ & \quad + \frac{\pi i}{4} \Sigma^{+-} (\Sigma^{++} - \Sigma^{--} - \Sigma^{-+}) n \delta' \\ & \quad - \frac{\pi i}{4} \Sigma^{+-} \Sigma^{-+} \delta'. \end{aligned} \quad (145)$$

But Σ^{+-} and Σ^{-+} are real and $(\Sigma^{--})^* = \Sigma^{++}$. Neglecting in (145) by real terms we find

$$\begin{aligned} & \sum_{i,j,k,l=\pm} (H_{\varepsilon R}^{ijkl})_1 \\ &= \left\{ \frac{\pi i}{4} \Sigma^{+-} \Sigma^{-+} (1+n) - \frac{\pi i}{4} \Sigma^{+-} \Sigma^{-+} n \right. \\ & \quad \left. - \frac{\pi i}{4} \Sigma^{+-} \Sigma^{-+} \right\} \delta' = 0. \end{aligned} \quad (146)$$

So the image part of divergences of Green function is equal to zero. Therefore Divergences of two-chain diagram can be subtracted by counterterms of asymptotical state.

14 Notes on Bogoliubov derivation of Boltzmann equations

In this section we study the problem of boundary conditions in Bogoliubov derivation of kinetic equations [4]. Let us consider N particle in \mathbb{R}^3 . Let q_i be

a coordinates of particle number i , and p_i be a momenta of particle number i , $i = 1, \dots, N$. Suppose that particles interacts by means the pair potential $V(q_i - q_j)$. We suppose that V belongs to the Schwartz space. Let $x_i = (p_i, q_i)$ be a point in the phase space Γ . Let $f(x_1, \dots, x_n)$ be a distribution function of N particle. If we want to point out that $f(x_1, \dots, x_N)$ depends on t we will write $f(x_1, \dots, x_N|t)$. Let

$$f_1(x_1) = \int dx_2 \dots dx_N f(x_1, \dots, x_N), \quad (147)$$

and

$$f_2(x_1, x_2) = \int dx_3, \dots, dx_N f(x_1, \dots, x_N) \quad (148)$$

be marginal distribution functions. Put by definition

$$\begin{aligned} \rho_1(x_1) &= N f_1(x_1), \\ \rho_2(x_1, x_2) &= N^2 f_2(x_1, x_2). \end{aligned} \quad (149)$$

If A is a function on the phase space Γ , $\Gamma = \mathbb{R}^{6N}$ and

$$A = \sum_{i=1}^N \mathcal{A}(x_i) \quad (150)$$

then

$$\begin{aligned} \langle A \rangle &= \int f(x_1, \dots, x_N) A(x_1, \dots, x_N) = \\ &= N \int dx_1 \mathcal{A}(x_1) f(x_1) = \\ &= \int dx \mathcal{A}(x) \rho_1(x). \end{aligned} \quad (151)$$

Now if A is a function on the phase space

$$A = \sum_{i \neq j} \mathcal{A}(x_i, x_j) \quad (152)$$

in the limit of large N we find

$$\langle A \rangle = \int dx_1 dx_2 \rho_2(x_1, x_2) \mathcal{A}(x_1, x_2). \quad (153)$$

Let us introduce also three-particle distribution function:

$$f_3(x_1, x_2, x_3) = \int dx_4 \dots dx_N f(x_1, \dots, x_N). \quad (154)$$

Let us derive the equation for $f(x)$. At first let us write equation of motion for $f(x_1, \dots, x_N)$. We have

$$\begin{aligned} \frac{\partial}{\partial t} f(x_1, \dots, x_N | t) + \sum_{i=1}^n \frac{p_i}{m} \nabla_i f(x_1, \dots, x_n | t) \\ - \sum_{i \neq j} \frac{\partial V(q_i - q_j)}{\partial q_i} \frac{\partial f(x_1, \dots, x_n | t)}{\partial p_i}. \end{aligned} \quad (155)$$

This equation is only an infinitesimal form of the Liouville theorem. Let us multiple this equation by N and integrate over dx_2, \dots, dx_N . Suppose that $f(x_1, \dots, x_N)$ is a function of rapid decay of momenta. This assumption admit us integrate over p_i by parts. We find:

$$\begin{aligned} \frac{\partial}{\partial t} \rho_1(x_1 | t) + \frac{p}{m} \nabla \rho_1(x_1, t) \\ + \int dx_2 \frac{p_2}{m} \frac{\partial}{\partial q_2} \rho_2(x_1, x_2 | t) \\ = \int dx_2 \frac{\partial V(q_1 - q_2)}{\partial q_1} \frac{\partial \rho_2(x_1, x_2 | t)}{\partial x_1}. \end{aligned} \quad (156)$$

Note that we kept here boundary term. Let us now talk about derivation of kinetic equation. According to a standard prescription we put $\rho_3(x_1, x_2, x_3) = 0$ in equation for $\rho_2(x_1, x_2)$. We find the following equation for ρ_2 :

$$\frac{d}{dt} \rho_2(x_1(t), x_2(t) | t) = 0, \quad (157)$$

where $(x_1(t), x_2(t))$ is a solution of corresponding two-body problem.

Condition of correlation breaking. In purpose of simplicity we consider only translation-invariant matter. Usual correlation-breaking condition has the form

$$\rho_2(x_1, x_2 | 0) = h(p'_1(x_1, x_2)) h(p'_2(x_1, x_2)). \quad (158)$$

Here h is a function on momenta-space of one particle. We consider only translation-invariant gas, so h depends only of momentum.

p_1 and p_2 are momenta of particles 1 and 2 at $t = -\infty$ if at $t = 0$ their coordinates and momenta was x_1 and x_2 respectively.

Proposition.

$$\frac{\partial}{\partial t}\rho_2(x_1, x_2) = 0. \quad (159)$$

Indeed, according to (157)

$$\rho_2(x_1, x_2|t) = \rho_2(x_1^0, x_2^0|0), \quad (160)$$

where x_1^0 and x_2^0 are phase coordinates of particles 1 and 2 respectively at a moment $t = 0$. Therefore

$$\rho_2(x_1, x_2|t) = h(p_1'(x_1^0, x_2^0))h(p_2'(x_1^0, x_2^0)). \quad (161)$$

But the points x_1^0 and x_2^0 come to the points x_1 and x_2 after the time t . So $(p_1'(x_1^0, x_2^0), p_2'(x_1^0, x_2^0)) = (p_1'(x_1, x_2), p_2'(x_1, x_2))$, and

$$\begin{aligned} \rho_2(x_1, x_2|t) &= h(p_1'(x_1^0, x_2^0))h(p_2'(x_1^0, x_2^0)) = \\ &= h(p_1'(x_1, x_2))h(p_2'(x_1, x_2)) = \rho_2(x_1, x_2|0). \end{aligned} \quad (162)$$

In result

$$\rho_2(x_1, x_2|t) = \rho_2(x_1, x_2|0). \quad (163)$$

The proposition is proved.

It follows from equations (157) and (159) that:

$$\begin{aligned} &(\frac{p_1}{m}\nabla_1 + \frac{p_2}{m}\nabla_2)f(x_1, x_2|t) \\ &= (\frac{\partial V(q_1 - q_2)}{\partial q_1}\frac{\partial}{\partial p_1} + \frac{\partial V(q_1 - q_2)}{\partial q_2}\frac{\partial}{\partial p_2})f(x_1, x_2|t). \end{aligned} \quad (164)$$

The function $h(p)$ can be found from the following equation

$$\rho_1(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \int \rho_2(x_1, x_2) dx_2. \quad (165)$$

But in zero order of gas parameter the particles are free and

$$\rho_1(x) = h(x). \quad (166)$$

Formula (164) is usually used for transformation of r.h.s. of equation (156) to the collision integral. From other hand the equation

$$\frac{\partial}{\partial t}\rho_2(x_1, x_2) = 0 \quad (167)$$

shows that there no irreversible evolution in the system. From other point of view we will show that the last term in the left hand side of (156) is equal to the scattering integral.

For simplicity we will show the case $v_1 = 0$, $v = \frac{p}{m}$. The general case can be reduced to this case by means of Galilei transformation. So let us consider the integral

$$I = \lim_{R \rightarrow \infty} \int_{V_R} d^3 p_2 \int dq_2 \frac{p_2}{m} \frac{\partial}{\partial q_2} \rho(0, 0, p_2, q_2), \quad (168)$$

where V_R is a ball of radius R with the center at zero. Let us integrate over dq_2 by using Gauss theorem. We find

$$I = \lim_{R \rightarrow \infty} \int d^3 p_2 \int_{S_R} dS \frac{p_2}{m} \cos \psi \rho(0, 0, p_2, q_2). \quad (169)$$

Here S_R is a boundary of V_R and ψ is an angle between to rays: first of them is parallel to p_2 , second starts from zero and comes throw q_2 . We have

$$I = \lim_{R \rightarrow \infty} \int d^3 p_2 \int_{S_R} dS \frac{p_2}{m} \cos \psi \times \\ h(p'_1(0, 0)) h(p'_2(p_2, q_2)). \quad (170)$$

Let us suppose that the particle scatters only then they are not too far from to each other. Then

$$h(p'_1(0, 0)) h(p'_2(p_2, q_2)) = \rho_1(p_2) \rho_1(0) \quad (171)$$

for all $q_2 \in S_R \setminus \mathcal{O}$, where \mathcal{O} is a small neighborhood of the point $q_0 := \frac{p_2}{|p_2|} R \in S_R$. Diameter of \mathcal{O} is approximately equal to diameter of $\text{supp} V$. Therefore the integral I is not equal to zero and equal to

$$I = \int d^3 p_2 \frac{p_2}{m} \int 2\pi b db \\ \times \{ \rho_1(p'_1((p_2, q_2(b)), (0, 0))) \rho_1(p'_2((p_2, q_2(b)), (0, 0))) \\ - \rho_1(p_2) \rho_1(0) \}, \quad (172)$$

where $b := q_2 - q_0$. But the right hand side of (172) is a usual collision integral.

Therefore if we keep boundary terms in BBGKI-chain we obtain the kinetic equations without collision integral.

15 Conclusion

In this paper we have studied the problem of divergences in Keldysh diagram technique which arise if the matter is non-equilibrium. We have considered some wide class of divergent diagrams and have proposed a method for renormalization of this divergences. We use this method for renormalization of one- and two-chain diagrams.

A general thesis that we want to illustrate in this series of papers consists in follows: to prove that the system tends to thermal equilibrium one should to take into account its behavior on its boundary. In the last section we have shown that some boundary terms in BBGKI-chain which are usually neglected in Bogoliubov derivation of kinetic equation compensate scattering integral in kinetic equation.

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